

INTEGRAL CONGRUENCE TWO HYPERBOLIC 5-MANIFOLDS

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ABSTRACT. In this paper, we classify all the orientable hyperbolic 5-manifolds that arise as a hyperbolic space form H^5/Γ where Γ is a torsion-free subgroup of minimal index of the congruence two subgroup Γ_2^5 of the group Γ^5 of positive units of the Lorentzian quadratic form $x_1^2 + \cdots + x_5^2 - x_6^2$. We also show that Γ_2^5 is a reflection group with respect to a 5-dimensional right-angled convex polytope in H^5 . As an application, we construct a hyperbolic 5-manifold of smallest known volume $7\zeta(3)/4$.

1. INTRODUCTION

According to Carl Ludwig Siegel,

*the mathematical universe is inhabited not only by important species
but also by interesting individuals.*

Arithmetic hyperbolic manifolds form an important species of hyperbolic manifolds. In this paper, we construct and classify some interesting individuals. These individuals are interesting because of their dimension, five, and the value, $28\zeta(3)$, of their volume. The volumes of our 5-manifolds are interesting because the value of the Riemann zeta function at three, $\zeta(3)$, is known to be an irrational number [11], in contrast to dimension three, where the irrationality of volumes of hyperbolic 3-manifolds is unknown.

Our manifolds are explicitly described by gluing together the sides of a hyperbolic convex polytope. These manifolds are the first examples of explicitly described complete hyperbolic 5-manifolds of finite volume. One of our examples has a group of symmetries of order 16 that acts freely on the manifold. The resulting quotient hyperbolic 5-manifold has volume, $7\zeta(3)/4$, which is the smallest known volume of a complete hyperbolic 5-manifold. The previous smallest known volume was $14\zeta(3)$ which is the volume of a hyperbolic 5-manifold constructed by B. Everitt [1]. A volume lower bound of R. Kellerhals [3] suggests that our quotient manifold has very small volume and may be a minimum volume open hyperbolic 5-manifold. See our paper [9], for the volumes of some arithmetic hyperbolic 5-manifolds.

We now set up notation. A real $(n+1) \times (n+1)$ matrix A is said to be *Lorentzian* if A preserves the *Lorentzian inner product*

$$x \circ y = x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1}.$$

The *hyperboloid model* of hyperbolic n -space is the metric space

$$H^n = \{x \in \mathbb{R}^{n+1} : x \circ x = -1 \text{ and } x_{n+1} > 0\}$$

with metric d defined by

$$\cosh d(x, y) = -x \circ y.$$

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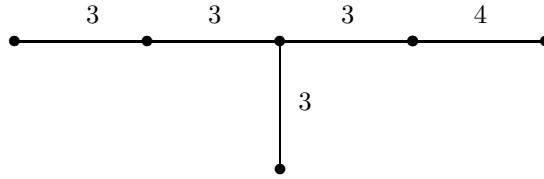
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A Lorentzian $(n+1) \times (n+1)$ matrix A is said to be either *positive* or *negative* according as A maps H^n to H^n or $-H^n$. The isometries of H^n correspond to the positive Lorentzian $(n+1) \times (n+1)$ matrices.

Let Γ^n be the group of positive Lorentzian $(n+1) \times (n+1)$ matrices with integer entries. The group Γ^n is an infinite discrete subgroup of the group $O(n, 1)$ of Lorentzian $(n+1) \times (n+1)$ matrices. The *principal congruence two subgroup* of Γ^n is the group Γ_2^n of all matrices in Γ^n that are congruent to the identity matrix modulo two. The congruence two subgroup Γ_2^n is not torsion-free, but it only has 2-torsion by Theorem IX.7 of Newman [5].

2. CONGRUENCE TWO SUBGROUP

In this section, we determine the structure of the congruence two subgroup Γ_2^5 of the group Γ^5 of integral, positive, Lorentzian 6×6 matrices. Before reading this section, we advise the reader to read the first four sections of our paper [10] which determines the structure of Γ_2^2 , Γ_2^3 , and Γ_2^4 . Vinberg [12] proved that the group Γ^5 is a reflection group with respect to a noncompact 5-simplex Δ^5 in H^5 whose Coxeter diagram is



The volume of Δ^5 is $7\zeta(3)/15360$, see [9]. Vertices for Δ^5 are

$$(0,0,0,0,0,1), (\sqrt{6}/6,\sqrt{6}/6,\sqrt{6}/6,0,0,\sqrt{6}/2), \\ (\sqrt{2}/2,\sqrt{2}/2,0,0,0,\sqrt{2}), (\sqrt{5}/5,\sqrt{5}/5,\sqrt{5}/5,\sqrt{5}/5,0,3\sqrt{5}/5), \\ (1/2,1/2,1/2,1/2,1/2,3/2), (1,0,0,0,0,1) \text{ (at infinity).}$$

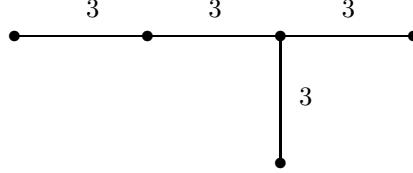
The group Γ^5 is generated by the following six matrices that represent the reflections of H^5 in the sides of Δ^5 ,

$$\left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right), \quad \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right), \quad \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right),$$

$$\left(\begin{array}{cccccc} 0 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 2 \end{array} \right), \quad \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right), \quad \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Let Σ^5 be the group generated the first five matrices in the above list of matrices. Then Σ^5 is the group generated by the reflections in the sides of Δ^5 incident with the vertex $(1/2, 1/2, 1/2, 1/2, 1/2, 3/2)$. Therefore Σ^5 is isomorphic to a spherical 4-simplex reflection group whose Coxeter diagram is obtained from the Coxeter

diagram of Γ_2^5 by deleting its right most vertex and its adjoining edge. Hence Σ^5 has the Coxeter diagram



The order of the Coxeter group Σ^5 is 1920. None of the elements of Σ^5 are in Γ_2^5 . By Lemma 16 of [9], the index of Γ_2^5 in Γ^5 is 1920. Hence Σ^5 is a set of coset representatives for Γ_2^5 in Γ^5 . We therefore have a natural, split, short, exact sequence of groups

$$1 \rightarrow \Gamma_2^5 \rightarrow \Gamma^5 \rightarrow \Sigma^5 \rightarrow 1.$$

Let $P^5 = \Sigma^5 \Delta^5$. Then P^5 is a convex hyperbolic polytope which is subdivided into 1920 copies of Δ^5 obtained by reflecting in the sides of Δ^5 that are incident with the vertex $(1/2, 1/2, 1/2, 1/2, 1/2, 3/2)$. The polytope P^5 has 16 actual vertices and 10 ideal vertices. The ideal vertices are

$$(1, 0, 0, 0, 0, 1), (0, 1, 0, 0, 0, 1), (0, 0, 1, 0, 0, 1), (0, 0, 0, 1, 0, 1), (0, 0, 0, 0, 1, 1), \\ (0, 1, 1, 1, 1, 2), (1, 0, 1, 1, 1, 2), (1, 1, 0, 1, 1, 2), (1, 1, 1, 0, 1, 2), (1, 1, 1, 1, 0, 2).$$

The 10 ideal vertices of P^5 are the vertices of a regular ideal 5-dimensional cross polytope R^5 centered at the point $(1/2, 1/2, 1/2, 1/2, 1/2, 3/2)$. The regular cross polytope R^5 has 32 sides each of which is a regular ideal 4-simplex.

Let e_1, \dots, e_6 be the standard basis of \mathbb{R}^6 and let T^5 be the 5-simplex in H^5 with one actual vertex at the origin e_6 of H^5 and five ideal vertices $e_i + e_6$ for $i = 1, \dots, 5$. We call T^5 a *corner 5-simplex*, with e_6 the *corner vertex* of T^5 , and each side of T^5 incident with e_6 a *corner side* of T^5 . The 5-simplex T^5 has five corner sides. The sixth side of T^5 is a regular ideal 4-simplex. The dihedral angle between any two corner sides of T^5 is a right angle. The dihedral angle between the ideal side of T^5 and a corner side is $\pi/3$. Thus T^5 is a hyperbolic Coxeter simplex.

The regular ideal cross polytope R^5 is subdivided into 32 copies of T^5 fitting together around their corner vertices at the point $(1/2, 1/2, 1/2, 1/2, 1/2, 3/2)$. Each side of R^5 is the ideal side of one of the 32 corner simplices subdividing R^5 . The dihedral angles between the ideal sides and a common corner side of two adjacent corner simplices subdividing R^5 add up to the dihedral angle $2\pi/3$ of R^5 .

The polytope P^5 is obtained from R^5 by gluing onto each of 16 mutually non-adjacent sides of R^5 a corner 5-simplex with one of the attached corner 5-simplices being T^5 . The corner vertices of the attached corner 5-simplices are the actual vertices of P^5 . The actual vertices of P^5 are

$$(0, 0, 0, 0, 0, 1), (0, 0, 1, 1, 1, 2), (0, 1, 0, 1, 1, 2), (0, 1, 1, 0, 1, 2), \\ (0, 1, 1, 1, 0, 2), (1, 0, 0, 1, 1, 2), (1, 0, 1, 0, 1, 2), (1, 0, 1, 1, 0, 2), \\ (1, 1, 0, 0, 1, 2), (1, 1, 0, 1, 0, 2), (1, 1, 1, 0, 0, 2), (1, 1, 1, 1, 2, 3), \\ (1, 1, 1, 2, 1, 3), (1, 1, 2, 1, 1, 3), (1, 2, 1, 1, 1, 3), (2, 1, 1, 1, 1, 3).$$

The polytope P^5 has 16 sides. The remaining 16 sides of R^5 extend to the 16 sides of P^5 . Each of these 16 sides of R^5 is a regular ideal 4-simplex which is coplanar with a corner side of five of the attached corner 5-simplices. Thus each

side of P^5 is obtained from a regular ideal 4-simplex S^4 by gluing on a corner 4-simplex onto each side of S^4 . Hence, each side of P^5 is congruent to the polytope P^4 described in our paper [10]. The group of symmetries of $P^5 = \Sigma^5 \Delta^5$ is Σ^5 which is a subgroup of index two of the group of symmetries of R^5 .

The dihedral angles of P^5 are the dihedral angles of the attached corner 5-simplices between corner sides, and so P^5 is a right-angled polytope. It is worth noting that L. Potyagailo and E. Vinberg [6] have proven that a 5-dimensional, right-angled, hyperbolic polytope of finite volume has at least 16 sides, and so P^5 is a 5-dimensional, right-angled, hyperbolic polytope of finite volume with the smallest possible number of sides.

The polytope P^5 has 16 sides, five of them are coplanar with the coordinate hyperplanes $x_i = 0$, for $i = 1, \dots, 5$. Let $D = \text{diag}(1, 1, 1, 1, -1, 1)$ be the last generator of Δ^5 . The diagonal matrix D represents the reflection of H^5 in the hyperplane $x_5 = 0$. The matrix D is in Γ_2^5 . The reflections of H^5 in the 16 sides of P^5 are represented by matrices of the form ADA^{-1} with A in Σ^5 . Each such matrix ADA^{-1} is in Γ_2^5 , since Γ_2^5 is a normal subgroup of Γ^5 . Now since Σ^5 is a set of coset representatives for Γ_2^5 in Γ^5 , we have that $P^5 = \Sigma^5 \Delta^5$ is a fundamental polytope for Γ_2^5 . We therefore have the following theorem.

Theorem 1. *The congruence two subgroup Γ_2^5 of the group Γ^5 of integral, positive, Lorentzian 6×6 matrices is a reflection group with respect to the right-angled polytope P^5 .*

We now describe the 16 matrices that represent the reflections of H^5 in the sides of P^5 . Five of the sides of P^5 are coplanar with the coordinate hyperplanes $x_i = 0$, for $i = 1, \dots, 5$. The reflections in the five coordinate hyperplanes are represented by diagonal matrices with diagonal entries all 1 except for a -1 in the i th diagonal slot. The polytope P^5 has 10 sides each of which is perpendicular to three of the coordinate hyperplanes in all 10 combinations of three coordinate hyperplanes. The matrices representing the reflections in these 10 sides are obtained from the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -2 & 2 \\ 0 & 0 & 0 & -2 & -1 & 2 \\ 0 & 0 & 0 & -2 & -2 & 3 \end{pmatrix}$$

by conjugating by a permutation matrix of the first five coordinates. Note that the three diagonal ones can lie in any one of the 10 combinations of three diagonal positions in the first five diagonal slots. The remaining side of P^5 is not incident with any of the coordinate hyperplanes. We call this side the *far side* of P^5 . The reflection of H^5 in the far side of P^5 is represented by the matrix

$$\begin{pmatrix} -1 & -2 & -2 & -2 & -2 & 4 \\ -2 & -1 & -2 & -2 & -2 & 4 \\ -2 & -2 & -1 & -2 & -2 & 4 \\ -2 & -2 & -2 & -1 & -2 & 4 \\ -2 & -2 & -2 & -2 & -1 & 4 \\ -4 & -4 & -4 & -4 & -4 & 9 \end{pmatrix}.$$

Let K^5 be the group of 32 diagonal matrices $\text{diag}(\pm 1, \pm 1, \pm, \pm 1, \pm 1, 1)$. Then K^5 is a subgroup of Γ_2^5 . The next corollary follows immediately from Theorem 1.

Corollary 1. *Every nonidentity element of Γ_2^5 of finite order has order two, every finite subgroup of Γ_2^5 is conjugate in Γ^5 to a subgroup of the elementary 2-group K^5 , and there are 16 conjugacy classes of maximal finite subgroups of Γ_2^5 in Γ_2^5 corresponding to the 16 actual vertices of P^5 .*

3. TORSION-FREE SUBGROUPS OF INDEX 32

Let Γ be a torsion-free subgroup of Γ_2^5 of finite index. Then the finite group K^5 acts freely on the set of cosets of Γ in Γ_2^5 by $\Gamma g \mapsto \Gamma gk$, since Γ is torsion-free. Therefore $|K^5| = 32$ divides $[\Gamma_2^5 : \Gamma]$.

Now suppose that $[\Gamma_2^5 : \Gamma] = 32$. Then the convex polytope $Q^5 = K^5 P^5$ is a fundamental polytope for Γ . The polytope is subdivided into 32 copies of P^5 , one of which is P^5 , that fit together around the origin e_6 of H^5 , with one copy of P^5 in each of the 32 orthants of H^5 bounded by five coordinate hyperplanes of H^5 .

The polytope Q^5 has 72 sides of two different types. A side of P^5 lying in a coordinate hyperplane is identified with a side of the copy of P^5 across the coordinate hyperplane. This eliminates $5 \cdot 32$ sides of the 32 copies of P^5 subdividing Q^5 . Let S be a side of P^5 that is perpendicular to three coordinate hyperplanes and let K_S be the subgroup of K^5 generated by the reflections in the three coordinate hyperplanes perpendicular to S . Then K_S leaves the hyperplane of H^5 spanned by S invariant. Hence the eight sides $\{kS : k \in K_S\}$ of the copies of P^5 subdividing Q^5 are coplanar. These eight copies of P^4 form a side $K_S S$ of Q^5 . The intersection of the eight copies of P^4 forming $K_S S$ is a hyperbolic line which we call the *axis* of the side $K_S S$ of Q^5 . The axis of the side $K_S S$ is an edge of S joining two ideal vertices $e_i + e_6$ and $e_j + e_6$ of S . There are a total of 40 sides of Q^5 of this type which accounts for $40 \cdot 8 = 10 \cdot 32$ sides of the 32 copies of P^5 subdividing Q^5 . Now K^5 maps the far side of P^5 to 32 sides of Q^5 that are congruent to P^4 . This accounts for all of the $16 \cdot 32$ sides of the 32 copies of P^5 subdividing Q^5 . Thus Q^5 has a total of 72 sides, 32 small sides congruent to P^4 , and 40 large sides made up of eight copies of P^4 glued together around a common infinite edge. The axes of the 40 large sides of Q^5 are the edges of the regular ideal 5-dimensional cross polytope O^5 with vertices $\pm e_i + e_6$, for $i = 1, \dots, 5$. There is one small side of Q^5 in each of the 32 orthants of H^5 bounded by five coordinate hyperplanes. The small sides of Q^5 are pairwise disjoint. The polytope Q^5 has the same symmetry group as the regular cross polytope O^5 . All the dihedral angles of the sides of Q^5 are right angles and all the dihedral angles of Q^5 are right angles.

Let S be a side of Q^5 and let r be the reflection of H^n in the side S . Then there is a reflection s in a side of P^5 and an element ℓ of K^5 such that $r = \ell s \ell^{-1}$. Hence r is in Γ_2^5 . Now since K^5 is a set of coset representatives for Γ in Γ_2^5 , there is an element g of Γ and an element k of K^5 such that $r = gk$. Then $g = rk$. Thus rk is a side-pairing transformation in Γ for Q^5 that pairs the side $S' = kS$ of Q^5 to the side S of Q^5 . Summarizing, we have the following lemma.

Lemma 1. *Let Γ be a torsion-free subgroup of Γ_2^5 of index 32. Then the right-angled polytope Q^5 is an exact fundamental polytope for Γ . The side-pairing transformations of Q^5 are of the form rk where k is in K^5 and r is the reflection in a side of Q^5 .*

4. GEOMETRY OF THE FUNDAMENTAL POLYTOPE Q^5

The polytope Q^5 has 160 actual vertices and 90 ideal vertices. The polytope P^4 has five actual vertices and five ideal vertices. All the actual vertices of Q^5 are vertices of small sides for a total of $5 \cdot 32 = 160$ actual vertices. The 160 actual vertices are obtained from $(\pm 2, \pm 1, \pm 1, \pm 1, \pm 1, 3)$ by permuting the first five coordinates. The ten ideal vertices $\pm e_i + e_6$ of the regular cross polytope O^5 are ideal vertices of Q^5 , which we call *large ideal vertices*. Each large ideal vertex is a vertex of eight large sides and no small sides. All the other ideal vertices of Q^5 are ideal vertices of small sides, with each ideal vertex shared by two small sides in adjacent orthants, for a total of $(5 \cdot 32)/2 = 80$ ideal vertices of small sides, which we call *small ideal vertices*. The 80 small ideal vertices are obtained from $(0, \pm 1, \pm 1, \pm 1, \pm 1, 2)$ by permuting the first five coordinates.

We next describe the ridges (3-faces) of Q^5 . The polytope Q^5 has 560 ridges of two different types. The polytope P^4 has 10 sides, each of which is congruent to the polyhedron P^3 described in our paper [10]. See Figure 1. The polyhedron P^3 has five vertices, two actual and three ideal, and six sides, each of which is an ideal right triangle congruent to P^2 . All the dihedral angles of P^3 are right angles. The polytope Q^5 has $10 \cdot 32 = 320$ small ridges congruent to P^3 . Each small ridge is a side of just one small side of Q^5 and just one large side of Q^5 .

Let S and T be adjacent sides of P^5 that are both perpendicular to three coordinate hyperplanes. These two sets of three coordinate hyperplanes have two hyperplanes in common. The intersection of S and T is a ridge R of P^5 , and so R is congruent to P^3 . Now R is perpendicular to the two common coordinate hyperplanes and lies in the other two coordinate hyperplanes. Let K_R be the subgroup of K^5 generated by the reflections in the two coordinate hyperplanes perpendicular to R . Then K_R leaves the 3-plane of H^5 spanned by R invariant. Hence the four ridges $\{kR : k \in K_R\}$ of copies of P^5 subdividing Q^5 are coplanar. These four copies of P^3 form a ridge $K_R R$ of Q^5 . The intersection of the four copies of P^3 forming $K_R R$ is a hyperbolic ray which we call the *axis* of the ridge $K_R R$ of Q^5 .

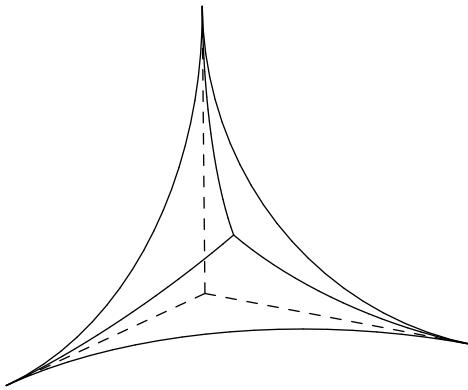
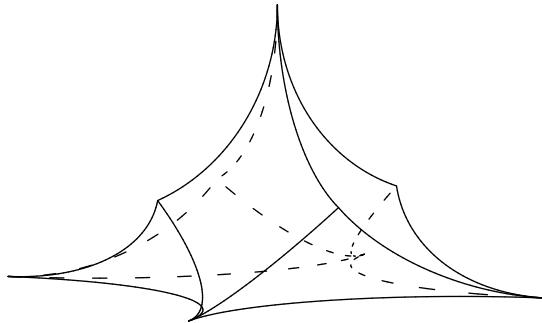


FIGURE 1. The polyhedron P^3

FIGURE 2. A large ridge of Q^5

The axis of the ridge $K_R R$ is an edge of R joining a large ideal vertex to an actual vertex of R . The polyhedron $K_R R$ is congruent to the top half of the rhombic dodecahedron Q^3 in Figure 5 of our paper [10]. See Figure 2. The polyhedron $K_R R$ has nine vertices, four actual and five ideal, and nine sides, four triangles congruent to P^2 , four rhombi made up of two copies of P^2 glued together along a line edge, and an ideal square congruent to Q^2 . All the dihedral angles of the polyhedron $K_R R$ are right angles. A large side of Q^5 has 20 sides, eight small sides in common with adjacent small sides of Q^5 and 12 large sides in common with adjacent large sides of Q^5 . Thus Q^5 has a total of $40 \cdot 8 = 320$ small ridges congruent to P^3 and $(40 \cdot 12)/2 = 240$ large ridges made up of four copies of P^3 glued together around a common ray edge.

Let v be a large ideal vertex of Q^5 and let T be a large side of Q^5 with ideal vertex v . Each large side of T , with ideal vertex v , is adjacent to four large sides of T , with ideal vertex v , along its four rhombic faces. Therefore the link of the ideal vertex v in T is a cube and T has six large sides with ideal vertex v . The polytope Q^5 has eight large sides with ideal vertex v . Each large side of Q^5 , with ideal vertex v , is adjacent to six large sides of Q^5 with ideal vertex v . Hence the link of the large ideal vertex v of Q^5 is a 4-cube.

Let T be a large side of Q^5 with large ideal vertices v and w . Then T has six large sides with ideal vertex v and six large sides with ideal vertex w . Each large side of T , with ideal vertex v , is adjacent to a large side of T , with ideal vertex w , along a common ideal square face. Each large side of T is adjacent to four small sides along its four triangle faces and each small side of T is adjacent to six large sides of T along its six triangle faces.

Let u be a small ideal vertex of Q^5 and let S be a small side of Q^5 with ideal vertex u . The link of u in S is a cube. Hence the small side S of Q^5 is adjacent to six large sides of Q^5 with ideal vertex u . Let R be a large ridge of Q^5 with ideal vertex u . Then u is a vertex of the ideal square face of R . The link of u in R is a rectangle made up of two adjacent squares. Let T be a large side of Q^5 with ideal vertex u . Then two adjacent large sides of T , with a common ideal square face, have u as an ideal vertex. Hence the link of u in T is a rectangular box made up of two adjacent cubes. Thus T has four large sides and two small sides with ideal vertex u . Finally, the link of the small ideal u in Q^5 is a rectangular 4-box made

up of two adjacent 4-cubes. Thus Q^5 has six large sides and two small sides with ideal vertex u . The large side T of Q^5 has $8 \cdot 2 = 16$ actual vertices, two large ideal vertices, and $(8 \cdot 3)/2 = 12$ small ideal vertices.

The polytope Q^5 has 1360 2-faces of three different types, triangles congruent to P^2 , rhombi made up of two copies of P^2 , and ideal squares. Each triangle face of Q^5 is a face of a small side of Q^5 . Now P^3 has six faces congruent to P^2 and P^4 has ten sides congruent to P^3 . Hence P^4 has 30 triangle faces. Therefore Q^5 has $32 \cdot 30 = 960$ triangle faces. Let F be a rhombic face of Q^5 . Then F is a face of a large ridge of Q^5 , and so F has a large ideal vertex v . Hence F corresponds to an edge in the link of v . The link of v is a 4-cube. A 4-cube has 32 edges. Therefore Q^5 has $10 \cdot 32 = 320$ rhombic faces. Let S be an ideal square face of Q^5 . Then S is a face of a large ridge R_0 of Q^5 with large ideal vertex v . Now R_0 is the intersection of two large sides T_1 and T_2 of Q^5 with large ideal vertices u, v and w , respectively. The ridge R_0 is adjacent to a large ridge R_i in T_i along S for $i = 1, 2$. The ridges R_1 and R_2 are adjacent along S in the large side T_3 with large ideal vertices u and w . Thus S is a face of exactly three large ridges in Q^5 . Therefore there are a total of $240/3 = 80$ ideal square faces of Q^5 .

The polytope Q^5 has 1120 edges of two different types, rays and lines. Every line edge of Q^5 is an edge of a small side of Q^5 . The line edges of P^4 are the edges of a regular ideal 4-simplex. Hence P^4 has 10 line edges. Therefore Q^5 has $32 \cdot 10 = 320$ line edges. The link of an actual vertex v of Q^5 is a right-angled spherical 4-simplex whose vertices correspond to the edges of Q^5 with vertex v . Thus Q^5 has $5 \cdot 160 = 800$ ray edges.

The Euclidean closure of Q^5 in the projective disk model of hyperbolic 5-space is a Euclidean polytope \overline{Q}^5 , and so the Euler characteristic of \overline{Q}^5 is one. As a consistency check, we compute $\chi(\overline{Q}^5)$ in terms of the face decomposition of \overline{Q}^5 ,

$$\chi(\overline{Q}^5) = 250 - 1120 + 1360 - 560 + 72 - 1 = 1.$$

5. INTEGRAL, CONGRUENCE TWO, ORIENTABLE, HYPERBOLIC 5-MANIFOLDS

In order to describe a torsion-free subgroup of Γ_2^5 of index 32, we have to construct a proper side-pairing of the polytope Q^5 with side-pairing transformation of the form in Lemma 1. See Sections 11.1 and 11.2 of Ratcliffe [7] for a discussion of complete gluing of hyperbolic n -manifolds. The normalized solid angle subtended by a point in the interior of a k -face of Q^5 is 2^{k-5} for each $k = 0, \dots, 5$, since Q^5 is right-angled. Hence, in a proper side-pairing of Q^5 , we must have 2^{5-k} points in each cycle of points in the interior of k -faces. Thus in a proper side-pairing of Q^5 , there are 2^{5-k} k -faces of a single type in each cycle of k -faces for each $k = 0, \dots, 5$. Hence in a proper side-pairing of Q^5 there are $160/32 = 5$ cycles of actual vertices, $800/16 = 50$ cycles of ray edges, $320/16 = 20$ cycles of line edges, $960/8 = 120$ cycles of triangle faces, $320/8 = 40$ cycles of rhombic faces, $80/8 = 10$ cycles of ideal squares, $320/4 = 80$ cycles of small ridges, $240/4 = 60$ cycles of large ridges, $32/2 = 16$ cycles of small sides, and $40/2 = 20$ cycles of large sides.

Let M be a complete hyperbolic 5-manifold obtained by gluing together pairs of sides of Q^5 by a proper side-pairing. Then the Euler characteristic of M is zero. Although the k -faces of Q^5 are noncompact for all $k > 0$, we can compute the Euler characteristic of M from the face-cycle decomposition of M , in the usual way, since the Euler characteristic of the link of each cusp of M is zero. As a consistency

check, we compute $\chi(M)$ in terms of the face-cycle decomposition of M ,

$$\chi(M) = 5 - 70 + 170 - 140 + 36 - 1 = 0.$$

By a computer search, we found that there are exactly 6,616,152 proper side-pairings of Q^5 , all of whose elements are orientation preserving, of the form described in Lemma 1. These side-pairings fall into 55,168 equivalence classes under equivalence by a symmetry of Q^5 . The classification of integral, congruence two, orientable, hyperbolic 5-manifolds of minimum volume is summarized in our next theorem.

Theorem 2. *There are, up to isometry, exactly 3607 orientable hyperbolic space-forms H^5/Γ where Γ is a torsion-free subgroup of minimal index in the congruence two subgroup Γ_2^5 of the group Γ^5 of integral, positive, Lorentzian 6×6 matrices. All these hyperbolic 5-manifolds have either 10 or 12 cusps, with only 26 of these manifolds having 12 cusps. All these hyperbolic 5-manifolds have volume $28\zeta(3)$.*

Proof. Each side-pairing of Q^5 induces an equivalence relation on the 90 ideal vertices of Q^5 . The equivalence classes are cycles. The cycle of a large ideal vertex $v = \pm e_i + e_6$ of Q^5 is either just itself or itself and its antipodal vertex, since an element of K^5 either fixes v or maps v to $-v$. It turns out that the 10 large ideal vertices of Q^5 fall into either 5 or 6 cycles of the form 2, 2, 2, 2, 2 or 1, 1, 2, 2, 2, 2. The remaining 80 ideal vertices of Q^5 fall into either 5 or 6 cycles of the form 16, 16, 16, 16, 16 or 8, 8, 16, 16, 16, 16. The possible vertex cycle structures are 2, 2, 2, 2, 2, 16, 16, 16, 16 and 1, 1, 2, 2, 2, 2, 8, 8, 16, 16, 16, 16. Thus all the integral, congruence two, orientable, hyperbolic 5-manifolds of minimum volume have either 10 or 12 cusps. Only 26 of the manifolds have 12 cusps.

Let M be a 10-cusped manifold. Its side-pairing has a vertex cycle structure 2, 2, 2, 2, 2, 16, 16, 16, 16, 16. Consider a maximum open cusp of M of vertex cycle order 2 represented by two antipodal large ideal vertices $\pm v$ of Q^5 . This maximum open cusp is represented by the two open horoballs based at the ideal vertices $\pm v$ whose boundaries are tangent at the center e_6 of Q^5 . Thus the boundary horosphere of this maximum open cusp is tangent to itself at the point represented by the origin e_6 of H^5 . For example, if $v = e_1 + e_6$, then the equations of the supporting hyperplanes of the horospheres are $\pm x_1 - x_6 = -1$ which have e_6 as their only common solution in H^5 .

The two horospheres based at $\pm v$ that are tangent at the origin also pass through 32 actual vertices of Q^5 , and these 32 vertices represent another self-tangency point of the maximum cusp. For example, if $v = e_1 + e_6$, then the 32 actual vertices are $(\pm 2, \pm 1, \pm 1, \pm 1, \pm 1, 3)$. These two horospheres are also tangent to 48 centers of ideal square faces of Q^5 , and these points represent 6 more self-tangency points of the maximum cusp. For example, if $v = e_1 + e_6$, then the 48 centers of ideal square faces are the 48 points obtained from $(\pm 1, 0, 0, \pm 1, \pm 1, 2)$ by permuting the middle four coordinates. Thus, the boundary of a maximum open cusp of M of vertex cycle order 2 is tangent to itself at 8 points.

The boundary of a maximum open cusp of M of vertex cycle order 16 is also tangent to itself at 8 points. For example, if the cycle of ideal vertices consists of the 16 ideal vertices $(0, \pm 1, \pm 1, \pm 1, \pm 1, 2)$, the maximum open cusp is represented by 16 open horoballs based at the ideal vertices $(0, \pm 1, \pm 1, \pm 1, \pm 1, 2)$ whose horosphere boundaries are supported by the hyperplanes whose equations are

$$\pm x_2 \pm x_3 \pm x_4 \pm x_5 - 2x_6 = -1.$$

Each of these horospheres is tangent to 4 adjacent horospheres. These $4 \cdot 16/2 = 32$ tangency points are the 32 points obtained from $(0, 0, \pm 1, \pm 1, \pm 1, 2)$ by permuting the middle four coordinates. These 32 points represent four self-tangency points of the maximum cusp. Each horosphere passes through 8 actual vertices of Q^5 . These $8 \cdot 16 = 128$ vertices are obtained from $(\pm 1, \pm 2, \pm 1, \pm 1, \pm 1, 3)$ by permuting the middle four coordinates. These 128 vertices represent another 4 self-tangency points of the maximum cusp.

All the self-tangency points of the maximum cusps of M consist of only 16 points. Each of these 16 points is a self-tangency point of the boundary of exactly 5 of the maximum cusps. Thus M has a set of 16 canonical points. The 16 canonical points of M are represented by the 16 actual vertices of the polytope P^5 .

Let M be a 12-cusped manifold. Its side-pairing has a vertex cycle structure $1, 1, 2, 2, 2, 2, 8, 8, 16, 16, 16, 16$. Consider a maximum open cusp of M of vertex cycle order 1 represented by a large ideal vertex v of Q^5 . This maximum open cusp is represented by the open horoball based at v whose boundary passes through the centers of the 24 large sides of Q^5 that are not incident with $\pm v$. These 24 points represent 12 self-tangency points of the boundary of the maximum open cusp. For example, if $v = e_1 + e_6$, the maximum open cusp is represented by the open horoball based at v whose horosphere boundary is supported by the hyperplane whose equation is $x_1 - x_6 = -\sqrt{2}$. The 24 centers of large sides that are not incident with $\pm v$ are obtained from $(0, \pm\sqrt{2}/2, \pm\sqrt{2}/2, 0, 0, \sqrt{2})$ by permuting the middle four coordinates. These 24 points represent 12 self-tangency points of the maximum cusp. Thus, the boundary of a maximum open cusp of M of vertex cycle order 1 is tangent to itself in at least 12 points.

The boundary of a maximum open cusp of M of vertex cycle order 8 is also tangent to itself in at least 12 points. For example, if the cycle of ideal vertices consists of the 8 ideal vertices $(0, \pm 1, \pm 1, \pm 1, \pm 1, 2)$, with an even number of minus signs, the maximum open cusp is represented by 8 open horoballs, based at these ideal vertices, whose horosphere boundaries are supported by the hyperplanes whose equations are

$$\pm x_2 \pm x_3 \pm x_4 \pm x_5 - 2x_6 = -\sqrt{2},$$

with an even number of minus signs. Each of these horospheres are tangent to 6 adjacent horospheres. These $6 \cdot 8/2 = 24$ tangency points are the 24 points obtained from $(0, \pm\sqrt{2}/2, \pm\sqrt{2}/2, 0, 0, \sqrt{2})$ by permuting the middle four coordinates. These 24 points are centers of large sides of Q^5 and represent 12 self-tangency points of the maximum cusp of M .

The boundary of a maximum open cusp of M of vertex cycle order 2 or 16 is tangent to itself at 8 points as in the 10 cusp case. Thus, the cusps of vertex cycle order 1 or 8 are intrinsically different from the cusps of vertex cycle order 2 or 16. In fact, the volume of a maximum cusp of vertex cycle order 1 or 8 is 16 whereas the volume of a maximum cusp of vertex cycle order 2 or 16 is 8. Thus, a 12-cusped manifold has 4 large maximum cusps and 8 small maximum cusps. Now an isometry between 12-cusped manifolds must map small maximum cusps to small maximum cusps, and so the self-tangency points of the boundaries of the small maximum cusps are canonical points of a 12-cusped manifold. There are a total of 16 canonical points of a 12-cusped manifold represented by the 16 actual vertices of the polytope P^5 .

Now let M be either a 10- or 12-cusped manifold. Each actual vertex of P^5 is the vertex of a right-angled corner. This suggests a cut and paste operation on Q^5 . Cut Q^5 along the 5 coordinate hyperplanes into 32 copies of the polytope P^5 . By reassembling a copy of Q^5 around a different corner of P^5 than the origin according to the gluing pattern of the side-pairing of Q^5 that glues up to M , we get possibly 16 different side-pairings of Q^5 that glue up to M . We call such a cut and paste operation an *inside-out operation* on M .

We have 55,168 equivalence classes of side-pairings of Q^5 under equivalence by a symmetry of Q^5 . Each of these side-pairings of Q^5 determines 16 side-pairings on Q^5 after inside-out operations on Q^5 that yield the same manifold. After comparing the new side-pairings with the old ones, up to symmetry of Q^5 , the number of manifolds is reduced to 3607.

Let $\phi : M \rightarrow M'$ be an isometry between two of these 10- or 12-cusped manifolds. Let c and c' be the point of M and M' , respectively, represented by the center e_6 . Now c is a canonical point of M , and so $\phi(c)$ is a canonical point of M' . By applying an inside-out operation to M' if $\phi(c) \neq c'$, we may assume that $\phi(c) = c'$. Let $\tilde{\phi} : H^5 \rightarrow H^5$ be the lift of ϕ such that $\tilde{\phi}(e_6) = e_6$. Let Γ and Γ' be the discrete groups generated by the side-pairings of Q^5 that glue up M and M' , respectively. Then Q^5 is the Dirichlet polytope for Γ and Γ' centered at e_6 by Lemma 1. As $\Gamma' = \tilde{\phi}\Gamma\tilde{\phi}^{-1}$, we have that $\tilde{\phi}$ maps the Dirichlet polytope for Γ centered at e_6 to the Dirichlet polytope of Γ' centered at e_6 . Therefore $\tilde{\phi}$ is a symmetry of Q^5 . As the 3607 manifolds have been classified up to symmetry of Q^5 , we deduce that $\tilde{\phi}$ is a symmetry of the side-pairing of Q^5 for M , and so $M = M'$. Thus the classification of these manifolds up to isometry is complete.

□

6. SIDE-PAIRING CODING

In this section, we describe the coding that we use to specify the side-pairings of Q^5 for the integral, congruence two, hyperbolic 5-manifolds of minimum volume. Reading this section is necessary only if the reader wants to reconstruct one of the manifolds in Tables 3, ..., 6.

We encode an element $\text{diag}(a_1, a_2, a_3, a_4, a_5, 1)$ of K^5 , where each $a_i = \pm 1$, by

$$\sum_{i=1}^5 \frac{(1-a_i)}{2} 2^{i-1}$$

expressed as a single base 32 digit $0, \dots, 9, A = 10, \dots, V = 31$.

Table 1 gives the Lorentz normal vector s_i of the i th side S_i of Q^5 for $i = 1, \dots, 72$. The side-pairing map that maps the side S'_i to the side S_i is of the form $r_i k_i$ where r_i is the reflection in side S_i and k_i is an element of K^5 such that $S'_i = k_i S_i$. We have necessarily in these manifolds that the large sides pair in groups of four so that $k_{4i-3} = k_{4i-2} = k_{4i-1} = k_{4i}$ for $i = 1, 2, \dots, 10$ and then the remaining 32 small sides all pair in a similar fashion with $k_{41} = k_{42} = \dots = k_{72}$. Thus it suffices to take

$$k_1 k_5 k_9 k_{13} k_{17} k_{21} k_{25} k_{29} k_{33} k_{37} k_{41}$$

as an eleven digit base 32 code for the side-pairing.

By permuting coordinates, a given manifold may have several different encodings. Moreover there is an inside-out operation that takes the Q^5 fundamental domain apart and rearranges the pieces to give another side-pairing of Q^5 for the same

$s_1 = (1, 1, 0, 0, 0, 1)$	$s_{37} = (0, 0, 0, 1, 1, 1)$
$s_2 = (-1, 1, 0, 0, 0, 1)$	$s_{38} = (0, 0, 0, -1, 1, 1)$
$s_3 = (1, -1, 0, 0, 0, 1)$	$s_{39} = (0, 0, 0, 1, -1, 1)$
$s_4 = (-1, -1, 0, 0, 0, 1)$	$s_{40} = (0, 0, 0, -1, -1, 1)$
$s_5 = (1, 0, 1, 0, 0, 1)$	$s_{41} = (1, 1, 1, 1, 1, 2)$
$s_6 = (-1, 0, 1, 0, 0, 1)$	$s_{42} = (-1, 1, 1, 1, 1, 2)$
$s_7 = (1, 0, -1, 0, 0, 1)$	$s_{43} = (1, -1, 1, 1, 1, 2)$
$s_8 = (-1, 0, -1, 0, 0, 1)$	$s_{44} = (-1, -1, 1, 1, 1, 2)$
$s_9 = (0, 1, 1, 0, 0, 1)$	$s_{45} = (1, 1, -1, 1, 1, 2)$
$s_{10} = (0, -1, 1, 0, 0, 1)$	$s_{46} = (-1, 1, -1, 1, 1, 2)$
$s_{11} = (0, 1, -1, 0, 0, 1)$	$s_{47} = (1, -1, -1, 1, 1, 2)$
$s_{12} = (0, -1, -1, 0, 0, 1)$	$s_{48} = (-1, -1, -1, 1, 1, 2)$
$s_{13} = (1, 0, 0, 1, 0, 1)$	$s_{49} = (1, 1, 1, -1, 1, 2)$
$s_{14} = (-1, 0, 0, 1, 0, 1)$	$s_{50} = (-1, 1, 1, -1, 1, 2)$
$s_{15} = (1, 0, 0, -1, 0, 1)$	$s_{51} = (1, -1, 1, -1, 1, 2)$
$s_{16} = (-1, 0, 0, -1, 0, 1)$	$s_{52} = (-1, -1, 1, -1, 1, 2)$
$s_{17} = (0, 1, 0, 1, 0, 1)$	$s_{53} = (1, 1, -1, -1, 1, 2)$
$s_{18} = (0, -1, 0, 1, 0, 1)$	$s_{54} = (-1, 1, -1, -1, 1, 2)$
$s_{19} = (0, 1, 0, -1, 0, 1)$	$s_{55} = (1, -1, -1, -1, 1, 2)$
$s_{20} = (0, -1, 0, -1, 0, 1)$	$s_{56} = (-1, -1, -1, -1, 1, 2)$
$s_{21} = (0, 0, 1, 1, 0, 1)$	$s_{57} = (1, 1, 1, 1, -1, 2)$
$s_{22} = (0, 0, -1, 1, 0, 1)$	$s_{58} = (-1, 1, 1, 1, -1, 2)$
$s_{23} = (0, 0, 1, -1, 0, 1)$	$s_{59} = (1, -1, 1, 1, -1, 2)$
$s_{24} = (0, 0, -1, -1, 0, 1)$	$s_{60} = (-1, -1, 1, 1, -1, 2)$
$s_{25} = (1, 0, 0, 0, 1, 1)$	$s_{61} = (1, 1, -1, 1, -1, 2)$
$s_{26} = (-1, 0, 0, 0, 1, 1)$	$s_{62} = (-1, 1, -1, 1, -1, 2)$
$s_{27} = (1, 0, 0, 0, -1, 1)$	$s_{63} = (1, -1, -1, 1, -1, 2)$
$s_{28} = (-1, 0, 0, 0, -1, 1)$	$s_{64} = (-1, -1, -1, 1, -1, 2)$
$s_{29} = (0, 1, 0, 0, 1, 1)$	$s_{65} = (1, 1, 1, -1, -1, 2)$
$s_{30} = (0, -1, 0, 0, 1, 1)$	$s_{66} = (-1, 1, 1, -1, -1, 2)$
$s_{31} = (0, 1, 0, 0, -1, 1)$	$s_{67} = (1, -1, 1, -1, -1, 2)$
$s_{32} = (0, -1, 0, 0, -1, 1)$	$s_{68} = (-1, -1, 1, -1, -1, 2)$
$s_{33} = (0, 0, 1, 0, 1, 1)$	$s_{69} = (1, 1, -1, -1, -1, 2)$
$s_{34} = (0, 0, -1, 0, 1, 1)$	$s_{70} = (-1, 1, -1, -1, -1, 2)$
$s_{35} = (0, 0, 1, 0, -1, 1)$	$s_{71} = (1, -1, -1, -1, -1, 2)$
$s_{36} = (0, 0, -1, 0, -1, 1)$	$s_{72} = (-1, -1, -1, -1, -1, 2)$

TABLE 1. The Lorentz normal vectors of the 72 sides of Q^5

manifold. We take only one representative coding for each manifold and count the number of symmetries of the manifold that give the same side-pairing (these are all of the possible symmetries of the manifold). Thus the last digit of the code represents how the small sides pair, and by a symmetry we may as well take k_{41} coded by 1, 7, or $V = 31$, i.e., the small sides map with $k_i = \text{diag}(-1, 1, 1, 1, 1, 1)$, $\text{diag}(-1, -1, -1, 1, 1, 1)$, or $\text{diag}(-1, -1, -1, -1, -1, 1)$.

Finally, it is worth noting that the form of the side-pairing maps described above implies that all the torsion-free subgroups of Γ_2^5 of minimal index are normal.

7. TABLES OF MANIFOLDS

Tables 3-6 list side-pairings and isometric invariants of some examples of integral, congruence two, hyperbolic 5-manifolds of minimum volume. In each table, N is the row number in the complete list of all integral, congruence two, hyperbolic 5-manifolds of minimum volume. The 12-cusped manifolds have row numbers $1, \dots, 26$. The column headed by SP lists the side-pairing for the manifold in a coded form that is explained in the previous section. The column headed by S lists the number of symmetries of the manifolds. All the manifolds have a subgroup of symmetries corresponding to K^5 . Therefore, the number of symmetries is a multiple of 32. The possible orders are 32, 64, 128, 160, 192, 256, 384, 512, 1536. We ordered the 10- and 12-cusped manifolds so that the number of symmetries S is nonincreasing.

The column headed by H_i lists the i -th homology groups of the manifolds with the three numbers a, b, c representing $\mathbb{Z}^a \oplus \mathbb{Z}_2^b \oplus \mathbb{Z}_4^c$ and the single entry a representing \mathbb{Z}^a . All the 12-cusped manifolds have their fourth homology group isomorphic to \mathbb{Z}^{11} . All the 10-cusped manifolds have their fourth homology group isomorphic to \mathbb{Z}^9 . The column headed by LT lists the link types of the links of the cusps of the manifolds. In Table 3, the four large cusp link types are listed last.

There are 27 orientable, closed, flat 4-manifolds up to homeomorphism according to J. Hillman [2] and R. Levine [4]. We denote these 27 manifolds by O_1^4, \dots, O_{27}^4 in the order of Table D in Levine's thesis [4]. We also denote manifolds O_1^4, \dots, O_{26}^4 by the letters A, \dots, Z . Generators for $\pi_1(O_i^4)$, as a group of affine transformations of \mathbb{R}^4 , are given by Levine in Tables C and D of his thesis [4]. The first homology and holonomy groups of the closed orientable flat 4-manifolds were computed by Levine and are listed in Table D of his thesis [4].

The fundamental group of the link of a cusp of any one of our 5-manifolds is represented by a parabolic subgroup of Γ_2^5 , and so is conjugate to a subgroup of a stabilizer of an ideal vertex of P^5 . The link of an ideal vertex of P^5 is a Euclidean 4-cube, and so the stabilizer of an ideal vertex of P^5 is the reflection group in the sides of a Euclidean 4-cube. The holonomy group of the reflection group in the sides of a Euclidean 4-cube with one vertex at the origin and sides on the four coordinate hyperplanes of \mathbb{R}^4 is an elementary 2-group of rank 4 generated by the reflections in the four coordinate hyperplanes of \mathbb{R}^4 . Thus, the holonomy groups of the links of the cusps of our 5-manifolds are elementary 2-groups. These groups have rank at most two by Levine's computation of holonomy groups. Thus, the holonomy groups of the links of the cusps of our 5-manifolds are elementary 2-groups of rank 0, 1, or 2. The closed orientable flat 4-manifolds, with an elementary 2 holonomy group, are the manifolds A, \dots, L .

We denote the six orientable, closed, flat 3-manifolds by O_1^3, \dots, O_6^3 in the order given by J. Wolf [13]. Each of the manifolds O_i^4 for $i = 1, \dots, 27$ is a mapping torus over O_j^3 for some $j = 1, \dots, 6$; moreover, if the first Betti number of O_i^4 is 1, then O_i^4 is a mapping torus over O_j^3 for just one j . See Hillman [2] for a discussion. We denote a nondirect product mapping torus over O_j^3 by $O_j^3 \rtimes S^1$. A mapping torus decomposition of O_i^4 , either $O_j^3 \times S^1$ or $O_j^3 \rtimes S^1$, corresponds, respectively, to either a direct product $\pi(O_j^3) \times \mathbb{Z}$ or a semidirect product $\pi(O_j^3) \rtimes \mathbb{Z}$ decomposition of $\pi_1(O_i^4)$.

	<i>LT</i>	<i>HT</i>	<i>H</i> ₁	<i>MT</i>	<i>HG</i>
A	O_1^4	\mathbb{Z}^4		$O_1^3 \times S^1$	1
B	O_2^4	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^2$		$O_2^3 \times S^1$	\mathbb{Z}_2
C	O_3^4	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$		$O_2^3 \rtimes S^1$	\mathbb{Z}_2
D	O_4^4	$\mathbb{Z} \oplus \mathbb{Z}_4^2$		$O_6^3 \times S^1$	\mathbb{Z}_2^2
E	O_5^4	$\mathbb{Z} \oplus \mathbb{Z}_2^3$		$O_2^3 \rtimes S^1$	\mathbb{Z}_2^2
F	O_6^4	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$		$O_2^3 \rtimes S^1$	\mathbb{Z}_2^2
G	O_7^4	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$		$O_2^3 \rtimes S^1$	\mathbb{Z}_2^2
H	O_8^4	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$		$O_6^3 \rtimes S^1$	\mathbb{Z}_2^2
I	O_9^4	$\mathbb{Z} \oplus \mathbb{Z}_2^2$		$O_6^3 \rtimes S^1$	\mathbb{Z}_2^2
J	O_{10}^4	$\mathbb{Z} \oplus \mathbb{Z}_2^2$		$O_6^3 \rtimes S^1$	\mathbb{Z}_2^2
K	O_{11}^4	$\mathbb{Z} \oplus \mathbb{Z}_2^2$		$O_2^3 \rtimes S^1$	\mathbb{Z}_2^2
L	O_{12}^4	$\mathbb{Z} \oplus \mathbb{Z}_4$		$O_2^3 \rtimes S^1$	\mathbb{Z}_2^2

TABLE 2. The closed orientable flat 4-manifolds with elementary 2 holonomy

Table 2 lists the homeomorphism type, first homology group, mapping torus type, and holonomy group of the manifolds A, \dots, L . The mapping torus types of the manifolds A, \dots, L were determined by expressing $\pi_1(O_i^4)$ as either a direct product $\pi(O_j^3) \times \mathbb{Z}$ or a semidirect product $\pi(O_j^3) \rtimes \mathbb{Z}$.

The manifolds A and B are determined by their first homology groups. The manifold C is determined by its first homology and holonomy groups. The manifolds D and E are determined by their first homology groups. The manifolds F and G have the same first homology and holonomy groups and mapping torus type. We distinguished these manifolds by their fundamental groups. The manifold G does not occur as a link type of one of our 5-manifolds. The manifold H is determined by its first homology and holonomy groups and mapping torus type. The manifolds I and J have the same first homology and holonomy groups and mapping torus type. We distinguished these manifolds by their fundamental groups. The manifold K is determined by its first homology and holonomy groups and mapping torus type. The manifold L is determined by its first homology and holonomy groups. The manifold L does not occur as a link type of one of our 5-manifolds.

The coordinate hyperplane cross-sections of the cusps of our 5-manifolds are totally geodesic hypersurfaces. The mapping torus types of the links of the cusps of our 5-manifolds were determined by finding a connected orientable cross-section with a connected complement. Such a cross-section determines a mapping torus decomposition of the link of a cusp, since the link of each large ideal vertex of Q^5 is a 4-cube and by an inside-out operation on Q^5 we can represent any cusp by a cycle of large ideal vertices of Q^5 .

<i>N</i>	<i>SP</i>	<i>S</i>	<i>H</i> ₁	<i>H</i> ₂				<i>H</i> ₃	<i>LT</i>
1	24B8DPGPDB1	1536	4 7 0	12	4	6	20	JJJJJJJJAAAA	
2	PVSE8BBGEQ7	512	0 10 1	12	10	0	24	BBBBCCCCBBBB	
3	1B4EDQVLEB7	256	2 9 0	4	8	8	14	HHHJJJJJB BBB	
4	1B4EDQQGEB7	128	2 9 0	4	8	8	14	HHJJJJJJBB BB	
5	1BLEPVQGEB7	128	1 10 0	6	8	5	17	CCJJJJKKBBBB	
6	1B4ESQQLEV7	128	1 10 0	2	12	6	13	BBJJJJKKEEEE	
7	1B4EDMJGEB7	64	1 10 0	6	6	6	17	HHHKKKKKBBBB	
8	1B4EDMQPEB7	64	1 10 0	4	8	6	15	HHHJKKKKEEEE	
9	1EDQ8BBGME7	64	1 10 0	2	9	8	13	JJJJKKKKB BEE	
10	1B4EDQJPEB7	64	1 10 0	0	10	9	11	HHHHJJJJEEEEE	
11	1EDB8MMLJE7	64	0 11 0	4	10	4	16	BCHKKKKKEEEE	
12	1B4EPMJGEB7	64	0 11 0	4	8	6	16	HHHHKKKKBBBB	
13	1BSEPVJGEB7	64	0 11 0	4	8	5	16	CCHHHHHKKEEEE	
14	1B4EDMMLEJ7	64	0 11 0	4	7	6	16	HHHHKKKKKB BEE	
15	1BLEPMJGEB7	64	0 11 0	4	6	7	16	HHHHKKKKBBBB	
16	1BSEPMQGEB7	64	0 11 0	3	14	3	15	BBCJJKKKEEEE	
17	1B4ESQMLEB7	64	0 11 0	1	8	8	13	HHHHHJKKEEEE	
18	1EDJ8MBLJE7	64	0 11 0	1	8	8	13	HJKKKKKKEEEE	
19	1B4EDMJPEB7	32	1 10 0	3	7	8	14	HHHHJJKKB BEE	
20	1B4EDMMSEB7	32	1 10 0	3	7	8	14	HHHJJJKKB BEE	
21	1BSEPMJGEB7	32	0 11 0	7	10	2	19	BBCCHHHKKB BEE	
22	174EDQQPEJV	32	0 11 0	3	10	5	15	CHHHHKKKEEEE	
23	1BLEPMQGEB7	32	0 11 0	3	9	6	15	CHHJJKKKB BEE	
24	1B4EDQQPEV7	32	0 11 0	2	8	7	14	CHHHJJJKKEEEE	
25	174EDQQGEJV	32	0 11 0	2	7	8	14	HHJJKKKKKB BEE	
26	1B4EPMMLQBV	32	0 11 0	1	8	8	13	HHJJJKKKKEEEE	

TABLE 3. The orientable, 12-cusped, minimum volume, integral, congruence 2, hyperbolic 5-manifolds

8. A NON-ORIENTABLE EXAMPLE

Our first example of a minimal volume, integral, congruence two, hyperbolic 5-manifold was a nonorientable manifold M constructed by Steven Tschantz in 1993 by gluing together the sides of the polytope Q^5 by hand. The side-pairing code for M is 2549A81IKGV. The manifold M is the nonorientable hyperbolic 5-manifold of volume $28\zeta(3)$ mentioned in Ratcliffe [8]. The manifold M has $H_1(M) = \mathbb{Z}^5 \oplus \mathbb{Z}_2^6$, $H_2(M) = \mathbb{Z}^{10} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^9$, $H_3(M) = \mathbb{Z}^6 \oplus \mathbb{Z}_2^9$, and $H_4(M) = 0$. The manifold M has 10 cusps all of which are nonorientable. Five of the links of the cusps have their first homology group equal to $\mathbb{Z}^3 \oplus \mathbb{Z}_2$ and five have their first homology group equal to $\mathbb{Z}^2 \oplus \mathbb{Z}_2$. The manifold M has at least 160 symmetries.

In our paper [10], we showed that there are 3 orientable and 10 nonorientable minimal volume, integral, congruence two, hyperbolic 3-manifolds and there are 22 orientable and 1149 nonorientable minimal volume, integral, congruence two, hyperbolic 4-manifolds. In this paper, we showed that there are 3607 orientable minimal volume, integral, congruence two, hyperbolic 5-manifolds, and so we expect that there are hundreds of thousands of nonorientable, minimal volume, integral,

<i>N</i>	<i>SP</i>	<i>S</i>	<i>H</i> ₁	<i>H</i> ₂				<i>H</i> ₃	<i>LT</i>
27	2B7JB47JG81	1536	4	7	0	6	7	9	12 AADDxDDDD
28	24B8DPPGDB1	384	4	7	0	6	6	8	12 AADDJJJJJJ
29	27BBJ4J7G81	384	4	7	0	6	5	9	12 AADDxDDJJ
30	24BBJ7J7G81	256	4	7	0	6	6	8	12 AADDJJJJJJ
31	1DJEDBS2EV7	256	2	9	0	10	8	1	18 AACCCCJJJJ
32	24BJS8BJSG1	256	2	9	0	10	7	2	18 AACCCCHHHH
33	27BS74BSG81	256	2	9	0	6	9	4	14 CCCCCCDDDD
34	1EDDJBV2SE7	256	2	9	0	6	8	4	14 CCCCCCIIII
35	DES12BJEDV7	256	2	9	0	6	4	8	14 AAIIIIJJJJ
36	1ESDEBV2DJ7	256	2	9	0	2	9	8	10 CCDDDDIIII
37	1DEESBJ2VD7	256	2	9	0	2	6	10	10 CCDDDDHHHH
38	BDV8DBGQSP7	256	0	11	0	10	6	2	20 AABBBBKKKK
39	BDV824GLJP7	256	0	10	1	10	5	3	20 AABBBBHHHH
40	DJQBJMS2487	192	0	11	0	3	4	8	13 ADIIIIJJ
41	DBQQMLMLDBV	160	0	11	0	5	6	4	15 FFFFFKKKKK
42	14EPB8L2MGV	160	0	11	0	0	6	9	10 EEEEHHHHHH
43	DMQSJ7BLPEV	160	0	11	0	0	6	9	10 IIIIIIIIII
44	1DJEDBJ2EG7	128	2	9	0	10	6	2	18 AACCCCHHJJ
45	1DJEDBG2EJ7	128	2	9	0	6	10	3	14 CCCCCCDDJJ
46	27BV74BVG81	128	2	9	0	6	8	5	14 CCDDDDFFFF
47	27BS74BGS81	128	2	9	0	6	8	4	14 BBCCCCDDHH
48	1EDDJBG2JE7	128	2	9	0	6	8	4	14 BBCCCCDDII
49	1EDDJBJ2GE7	128	2	9	0	6	8	4	14 BBCCCCHHII
50	1DJEDBV2ES7	128	2	9	0	6	8	4	14 CCCCCCIIJJ
51	1DVEDBG2EV7	128	2	9	0	6	7	6	14 AADDFFFFII
52	27BBS4S7G81	128	2	9	0	6	7	5	14 BBDDFFFFII
53	1EDDVBV2GE7	128	2	9	0	6	7	5	14 BBFFFFIIJJ
54	24BBJSJSG81	128	2	9	0	6	6	6	14 AAFFFFHHII
55	1DVEDBV2EG7	128	2	9	0	6	6	6	14 AAFFFFIIII
56	1EDDVBG2VE7	128	2	9	0	6	6	6	14 CCDDFFFFJJ

TABLE 4. The first 60 orientable, 10-cusped, minimum volume, integral, congruence 2, hyperbolic 5-manifolds

congruence two, hyperbolic 5-manifolds. These nonorientable manifolds can be classified in the same way as the orientable manifolds, but the classification is beyond the reach of present computer technology.

9. A SMALL HYPERBOLIC 5-MANIFOLD

In this section, we describe a hyperbolic 5-manifold, of volume $7\zeta(3)/4$, that is obtained as a quotient space of one of our hyperbolic 5-manifolds. Let M be manifold number 27 which is the 10-cusped hyperbolic 5-manifold with the largest symmetry group listed first in Table 4. The group of symmetries of M has a subgroup of 16 orientation preserving symmetries that acts freely on M . This group of order 16 is generated by two elements α and β subject to the relations

<i>N</i>	<i>SP</i>	<i>S</i>	<i>H</i> ₁	<i>H</i> ₂			<i>H</i> ₃	<i>LT</i>
57	2B4JBDDJG81	128	2 9 0	6	5	7	14	AADDFFFFHH
58	DES12BSEDG7	128	2 9 0	6	4	8	14	AAIIJJJJJJ
59	1EJDEBG2DJ7	128	2 9 0	2	10	7	10	CCDDFFFFJJ
60	1DVEDBS2EJ7	128	2 9 0	2	10	7	10	CCFFFFIIJJ
61	1ESDEBG2DS7	128	2 9 0	2	8	9	10	CCDDDDDDII
62	1DEEJBG2JD7	128	2 9 0	2	8	8	10	BBDDFFFFHH
63	1DEEJBJ2GD7	128	2 9 0	2	8	8	10	BBFFFFHHHH
64	1EJDEBJ2DG7	128	2 9 0	2	8	8	10	BBFFFFHHJJ
65	1EDDVBS2JE7	128	2 9 0	2	8	8	10	BBFFFFJJJJ
66	1DEEJBV2SD7	128	2 9 0	2	8	8	10	CCFFFFHHII
67	1DEEJBS2VD7	128	2 9 0	2	8	8	10	CCFFFFHHJJ
68	1DEESBV2JD7	128	2 9 0	2	7	9	10	BBDDDDHHII
69	1DEESBS2GD7	128	2 9 0	2	7	9	10	BBDDDDHHJJ
70	1ESDEBS2DG7	128	2 9 0	2	7	9	10	BBDDDDIIJJ
71	1B4EDQGQEB7	128	2 9 0	2	6	10	10	BBDDHHJJJJ
72	DES12BGEDS7	128	2 9 0	2	6	10	10	BBDDIIJJJJ
73	1B4EDQLVEB7	128	2 9 0	2	6	10	10	BBHHIIJJJJ
74	1DEESBG2SD7	128	2 9 0	2	6	10	10	CCDDDDDDHH
75	1SV8MBBLEV7	128	1 10 0	8	7	2	17	BCCCCCCCCK
76	BDQED8PE4S7	128	1 10 0	8	7	2	17	BCCCCCCCCK
77	BDVE8BG2EV7	128	1 10 0	4	8	5	13	BCCFFFFII
78	BDVE8BV2EG7	128	1 10 0	4	8	5	13	BCCFFFFII
79	14BBMEJGM8V	128	0 11 0	10	8	1	20	AABBBCHHHH
80	1477QEJGQ8V	128	0 11 0	10	8	0	20	BBBBBCKKKK
81	BLQQ2S1L487	128	0 11 0	10	6	2	20	AABCCHHHH
82	1V48BMBMGV7	128	0 11 0	8	6	2	18	BCCCCCHHHH
83	BDVESQGLJV7	128	0 11 0	6	12	1	16	BBBBCJJJJ
84	1B48MSPBMG7	128	0 11 0	6	12	1	16	BCEEEKKKK
85	1V4JDMSPVPG7	128	0 11 0	6	11	2	16	BBBCCCKKKK
86	BDQESPMVLJ7	128	0 11 0	2	8	6	12	BBIIIIJJJJ

TABLE 5. The first 60 orientable, 10-cusped, minimum volume, integral, congruence 2, hyperbolic 5-manifolds (cont.)

$\alpha^8 = 1$, $\beta^2 = 1$, $\beta\alpha\beta = \alpha^3$. Lifts to H^5 of the isometries α and β of M are represented by the Lorentzian matrices

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 0 & 2 \end{array} \right), \left(\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & -1 & 2 \end{array} \right).$$

Let N be the quotient manifold under the action of this group of order 16. Then N is an orientable hyperbolic 5-manifold with two cusps. The cusps of N have type D and P . We have $H_1(P) = \mathbb{Z} \oplus \mathbb{Z}_4$. The mapping torus type of P is $O_6^3 \rtimes S^1$.

<i>N</i>	<i>SP</i>	<i>S</i>	<i>H</i> ₁	<i>H</i> ₂	<i>H</i> ₃	<i>LT</i>
100	1BS8JMG24B7	64	2 9 0	4 6 7	12	BBBCFFHII
200	14EEPBSBLD7	64	1 10 0	2 6 8	11	CCEEHHHHJJ
300	1B48BDQGVP7	32	2 9 0	8 7 4	16	AABCEHIKKK
400	14B8JLJGDB7	32	2 9 0	5 7 6	13	ABCHEHIKKK
500	14BB2LPQD87	32	2 9 0	3 6 8	11	BBCDFHHIJK
600	1BSSE4V2EG7	32	1 10 0	6 6 4	15	BCCDFHKKKK
700	14QDVSMED87	32	1 10 0	5 7 5	14	BBCCEEHIKK
800	1BEEP4S2JB7	32	1 10 0	5 6 5	14	CCCFFIJKKK
900	1BL82VLVQB7	32	1 10 0	4 8 5	13	BCCCDFFIIK
1000	1427D8GQDSV	32	1 10 0	4 6 6	13	BBCCFHHIIK
1100	17DDMEMEQ8V	32	1 10 0	4 6 6	13	BCCDEIIKK
1200	1DE82PQPSB7	32	1 10 0	4 6 6	13	CCCCEEHIIJ
1300	14EESB7QMGV	32	1 10 0	4 5 7	13	BCCEEHIIKK
1400	1DJ8V4SMLP7	32	1 10 0	3 7 7	12	BCCEIIKKKK
1500	14Q8JDQGVB7	32	1 10 0	3 6 7	12	BBFHHHHIIK
1600	1BM8BEPJSQ7	32	1 10 0	3 6 7	12	BCDEHHIIJK
1700	14B7E8M7LQV	32	1 10 0	3 6 7	12	CCCEEIIIJK
1800	14EJDBBLVE7	32	1 10 0	2 8 7	11	BBDEHHIIIJ
1900	17DE74QEJPV	32	1 10 0	2 6 8	11	BCDEFFIIKK
2000	17BJSES7L8V	32	1 10 0	2 6 8	11	BDEFHHIIJK
2100	14EESBLBGD7	32	1 10 0	2 6 8	11	CDEFFHHIKK
2200	1DQ8JMS24B7	32	1 10 0	1 6 9	10	BEEHIIIIKK
2300	14QDV8VLGS7	32	0 11 0	5 7 4	15	BCEEFHHIKK
2400	1BSEJMJ2ES7	32	0 11 0	4 7 5	14	BCCFFFHHJ
2500	14BPM8BPMQV	32	0 11 0	4 6 5	14	CCCDFFHIIK
2600	14QD8MPVQ7	32	0 11 0	3 8 5	13	BCCFFHIJKK
2700	14EEJLLBGD7	32	0 11 0	3 6 6	13	BCCFFHHIKK
2800	14BSJDJEMG7	32	0 11 0	3 6 6	13	CCCEFHHIIK
2900	17BQD4MSQBV	32	0 11 0	3 6 6	13	CEEFHHIIKK
3000	14EBMS7SPGV	32	0 11 0	2 6 7	12	BBDFHIIIJK
3100	1DV82BQLEP7	32	0 11 0	2 6 7	12	BCFHIIIIKKK
3200	1BVEDMV2PG7	32	0 11 0	2 6 7	12	CCFFFHIJJK
3300	1B7JQ4SJPEV	32	0 11 0	2 6 7	12	CFFHHIIIIII
3400	14EJPBP2GM7	32	0 11 0	1 6 8	11	CDDDFHHIJ
3500	1BMDBLMPQ8V	32	0 11 0	1 6 8	11	CFFFHHIIIK
3600	17BQM4BQLPV	32	0 11 0	0 6 9	10	FFFFHHHHIK

TABLE 6. A selection of the other orientable, 10-cusped, minimum volume, integral, congruence 2, hyperbolic 5-manifolds

The holonomy group of P is a dihedral group of order eight. The manifold P is determined by its first homology group and its mapping torus type. The cusp of N of type D is covered by the two cusps of M of type A while the cusp of N of type P is covered by the 8 cusps of M of type D . The manifold N has $H_1(N) = \mathbb{Z} \oplus \mathbb{Z}_4$, $H_2(N) = \mathbb{Z}_4^2$, $H_3(N) = \mathbb{Z}$, $H_4(N) = \mathbb{Z}$.

$s_1 = (-1, 0, 0, 0, 0, 0)$	$s_{17} = (-1, 0, 0, 0, 0, 0)$
$s_2 = (0, -1, 0, 0, 0, 0)$	$s_{18} = (0, 1, 0, 0, 0, 0)$
$s_3 = (0, 0, -1, 0, 0, 0)$	$s_{19} = (0, 0, -1, 0, 0, 0)$
$s_4 = (0, 0, 0, -1, 0, 0)$	$s_{20} = (0, 0, 0, -1, 0, 0)$
$s_5 = (0, 0, 0, 0, -1, 0)$	$s_{21} = (0, 0, 0, 0, -1, 0)$
$s_6 = (0, 0, 0, 1, 1, 1)$	$s_{22} = (0, 0, 0, 1, 1, 1)$
$s_7 = (0, 0, 1, 0, 1, 1)$	$s_{23} = (0, 0, 1, 0, 1, 1)$
$s_8 = (0, 0, 1, 1, 0, 1)$	$s_{24} = (0, 0, 1, 1, 0, 1)$
$s_9 = (0, 1, 0, 0, 1, 1)$	$s_{25} = (0, -1, 0, 0, 1, 1)$
$s_{10} = (0, 1, 0, 1, 0, 1)$	$s_{26} = (0, -1, 0, 1, 0, 1)$
$s_{11} = (0, 1, 1, 0, 0, 1)$	$s_{27} = (0, -1, 1, 0, 0, 1)$
$s_{12} = (1, 0, 0, 0, 1, 1)$	$s_{28} = (1, 0, 0, 0, 0, 1, 1)$
$s_{13} = (1, 0, 0, 1, 0, 1)$	$s_{29} = (1, 0, 0, 1, 0, 0, 1)$
$s_{14} = (1, 0, 1, 0, 0, 1)$	$s_{30} = (1, 0, 1, 0, 0, 0, 1)$
$s_{15} = (1, 1, 0, 0, 0, 1)$	$s_{31} = (1, -1, 0, 0, 0, 0, 1)$
$s_{16} = (1, 1, 1, 1, 1, 2)$	$s_{32} = (1, -1, 1, 1, 1, 2)$

TABLE 7. The Lorentz normal vectors of the 32 sides of P^5 and $\rho(P^5)$

The polytope Q^5 is subdivided into 32 copies of the polytope P^5 by the five coordinate hyperplanes $x_i = 0$ for $i = 1, \dots, 5$, and so the manifold M can be subdivided into 32 copies of P^5 . The subgroup of symmetries of order 16 of M whose orbit space is N acts freely on the 32 copies of P^5 subdividing M , and so N can be described by gluing together two copies of P^5 .

Let ρ be the reflection in the second coordinate hyperplane $x_2 = 0$. Then the polytopes P^5 and $\rho(P^5)$ are adjacent along a common side. We shall describe a side-pairing for the two polytopes P^5 and $\rho(P^5)$ that glues up to the hyperbolic 5-manifold N . Table 7 lists the Lorentz normal vectors of the sides of P^5 and $\rho(P^5)$.

We order the sides of P^5 and $\rho(P^5)$ by the ordering in Table 7. The i th side of $\{P^5, \rho(P^5)\}$ is paired to the j th side of $\{P^5, \rho(P^5)\}$ for $(i, j) = (1, 13), (2, 18), (3, 8), (4, 6), (5, 20), (7, 22), (9, 16), (10, 19), (11, 25), (12, 29), (14, 24), (15, 31), (17, 30), (21, 23), (26, 32), (27, 28)$. The Lorentzian matrices that pair the i th side to the j th side are given below. The second matrix is the identity matrix because side 2 of P^5 is equal to side 18 of $\rho(P^5)$.

$$\begin{aligned} & \left(\begin{array}{cccccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 2 \end{array} \right), \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right), \\ & \left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{array} \right), \left(\begin{array}{ccccc} -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 \end{array} \right), \end{aligned}$$

$$\begin{aligned}
& \left(\begin{array}{cccccc} 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & -1 & 0 & 2 \end{array} \right), \left(\begin{array}{cccccc} 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 2 \\ 0 & -1 & -2 & -1 & -2 & 3 \\ 0 & -1 & -3 & -1 & -2 & 4 \end{array} \right), \\
& \left(\begin{array}{cccccc} -1 & -2 & 0 & -1 & -2 & 3 \\ 0 & -2 & 0 & 0 & -1 & 2 \\ -1 & -2 & -1 & 0 & -2 & 3 \\ 0 & -1 & 0 & 0 & -2 & 2 \\ 0 & -2 & -1 & -1 & -2 & 3 \\ -1 & -4 & -1 & -1 & -4 & 6 \end{array} \right), \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 & 2 \end{array} \right), \\
& \left(\begin{array}{cccccc} 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 2 & 1 & 1 & -3 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 2 \\ 0 & -3 & -2 & -1 & -1 & 4 \end{array} \right), \left(\begin{array}{cccccc} -2 & -1 & 0 & -1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & -1 & -3 & 4 \end{array} \right), \\
& \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & -1 \\ -2 & -1 & -2 & -1 & 0 & 3 \\ -2 & 0 & -1 & 0 & 0 & 2 \\ -1 & -1 & 0 & 0 & 0 & 1 \\ -3 & -1 & -2 & -1 & 0 & 4 \end{array} \right), \left(\begin{array}{cccccc} -1 & -2 & 0 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 & 0 & 3 \end{array} \right), \\
& \left(\begin{array}{cccccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 2 \end{array} \right), \left(\begin{array}{cccccc} 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & -1 & 1 & 2 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ -1 & 1 & -2 & -1 & 1 & 3 \end{array} \right), \\
& \left(\begin{array}{cccccc} -1 & 2 & -1 & -2 & 0 & 3 \\ 0 & -2 & 0 & 1 & 0 & -2 \\ 0 & 2 & -1 & -2 & -1 & 3 \\ -1 & 2 & 0 & -2 & -1 & 3 \\ 0 & 1 & 0 & -2 & 0 & 2 \\ -1 & 4 & -1 & -4 & -1 & 6 \end{array} \right), \left(\begin{array}{cccccc} 0 & 1 & -2 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 2 & -2 & -1 & -1 & 3 \\ 0 & 2 & -3 & -1 & -1 & 4 \end{array} \right).
\end{aligned}$$

The set $P = P^5 \cup \rho(P^5)$ is a right-angled convex polytope with 20 sides. The above side-pairing of $\{P^5, \rho(P^5)\}$ determines a facet-pairing of P in the sense of §11.1 of [7] so that P is an inexact fundamental polytope for the discrete group Γ generated by the above 16 matrices. The hyperbolic manifold N is isometric to the space form H^5/Γ .

All the entries of the generators of Γ are integers, and so Γ is a torsion-free subgroup of Γ^5 . The volume of N is $28\zeta(3)/16 = 7\zeta(3)/4$. The volume of the fundamental domain Δ^5 of Γ^5 is $7\zeta(3)/15360$, and so the index of Γ in Γ^5 is $15360/4 = 3840$. The spherical Coxeter group $(3, 3, 3, 4)$ has order $2^5 \cdot 5! = 3840$, and so Γ is a torsion-free subgroup of Γ^5 of minimal index. Thus N is a minimal volume, integral, hyperbolic 5-manifold.

R. Kellerhals [3] has proved, by a horoball packing argument, that if M is an m -cusped hyperbolic 5-manifold, then

$$\text{vol}(M)/m > 0.3922.$$

Now N has two cusps and

$$\text{vol}(N)/2 = 1.0518\dots$$

This suggests that N has very small volume, and perhaps N is a minimal volume open hyperbolic 5-manifold.

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